Algebraic Surfaces of which every Plane-Section is Unicursal in the Light of n-Dimensional Geometry.

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Picard, in a recent memoir,* established the following theorem :

Les seules surfaces algébriques dont toutes les sections planes sont unicursales, sont les surfaces réglées unicursales et la surface du quatrième degré de Steiner.

In the present article I wish to give another proof of the same theorem, and to develop several allied propositions in the geometry of n-dimensions.

Picard notices at once that a surface of the kind under consideration, viz., of which every plane-section is unicursal, must be itself unicursal, and, accordingly, that there is a 1.1 correspondence between a point of the surface and a point of a plane (determined respectively by the homogeneous coordinate-sets (x, y, t, u), (α, β, γ)) defined by the equations

(1) $\quad x = f_1(\alpha, \beta, \gamma),\ y = f_2(\alpha, \beta, \gamma),\ z = f_3(\alpha, \beta, \gamma),\ t = f_4(\alpha, \beta, \gamma),$

where the f are integral homogeneous functions of α, β, γ of, say, degree n.

It is shown, l. c., pp. 77, 78, that there is no loss of generality in assuming that all the multiple points common to the triply-infinite system of curves

(2) $\quad Af_1(\alpha, \beta, \gamma) + Bf_2(\alpha, \beta, \gamma) + Cf_3(\alpha, \beta, \gamma) + Df_4(\alpha, \beta, \gamma) = 0$

are ordinary multiple points, and that in these points the curves have no common tangents. Let x_k be the number of k-ple points common to the f-curve-system.

To every curve of the system of curves (2) corresponds, in virtue of (1), the plane-section of the surface lying in the corresponding plane of the triply-infinite system of planes

(3) $$Ax + By + Cz + Dt = 0.$$

Every plane-section is unicursal; so every curve of system (2) must be unicursal.

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* "Sur les surfaces algébriques dont toutes les sections planes sont unicursales" (Kronecker's Journal d. Math., 1886, C, pp. 71–78).

If N is the degree of the surface, two planes intersect in a straight line which meets the surface in N points; any two plane-sections intersect in N points; so any two curves of the system must intersect in N points (distinct from the common base-point system common to all the curves). In this way we have the intersection- and unicursality-equations

$$\sum_k k^2 x_k = n^2 - N, \tag{4}$$

$$\sum_k \frac{1}{2} k(k-1) x_k = \frac{1}{2}(n-1)(n-2), \tag{5}$$

where in (5) the justifiable assumption is made that the arbitrary curve of the system (2) has no multiple point outside of the common base-point system. Subtracting, we have

$$\sum_k \frac{1}{2} k(k+1) x_k = \frac{1}{2} n(n+3) - (N+1). \tag{6}$$

In order that a point should be a k-ple point on a curve, $\frac{1}{2} k(k+1)$ conditions must be imposed on the curve. A curve of order n is determined by $\frac{1}{2} n(n+3)$ conditions. So from (6) the curves of order n passing through the common base-points of the system (2) contain $N+1$ arbitrary parameters; that is, are determined by $N+2$ linearly independent curves C^n of the system.

Here we cease to follow Picard, and notice that *a surface of order N of which every plane-section is unicursal may be regarded as the projection of a two-dimensional surface of order N in a flat space of $N+1$ dimensions.*

After Clifford, for *two-dimensional surface* we shall say *two-spread*, and for *flat space of $N+1$ dimensions* simply $(N+1)$-*flat.*

Let $y_1, y_2, \ldots \, y_{N+2}$ be the homogeneous coordinates of a point in an $(N+1)$-flat R_{N+1}. Then the equations

$$y_\kappa = f_\kappa(\alpha, \beta, \gamma), \qquad (\kappa = 1, 2, \ldots N+2), \tag{7}$$

where $f_\kappa(\alpha, \beta, \gamma) = 0$ for $\kappa = 1, 2, \ldots N+2$ are the equations of any $N+2$ linearly independent curves of the system C^n, of which the first four are, however, the f_1, f_2, f_3, f_4 of the preceding investigation, give the point-point representation on the (α, β, γ)-plane of a certain two-spread in the $(N+1)$-flat R_{N+1}. This two-spread is met by the $(N-1)$-flat of intersection of the two $(N+1)$-flats

$$\sum_\kappa a'_\kappa y_\kappa = 0, \quad \sum_\kappa a''_\kappa y_\kappa = 0$$

in a number of points equal to the order of the two-spread; these points corre-

spond to the N points of intersection of the two curves of the system

$$\sum_\kappa a'_\kappa f_\kappa(\alpha, \beta, \gamma) = 0, \quad \sum_\kappa a''_\kappa f_\kappa(\alpha, \beta, \gamma) = 0.$$

Hence the two-spread is of order N.

The equations (7) show at once that the two-spread may be projected from the $(N-3)$-flat passing through the coordinate vertices $5, 6, \ldots. N+2$ on the 3-flat $y_5 = y_6 = \ldots. = y_{N+2} = 0$ into the surface under consideration,

$$y_1 = f_1(\alpha, \beta, \gamma), \quad y_2 = f_2(\alpha, \beta, \gamma), \quad y_3 = f_3(\alpha, \beta, \gamma), \quad y_4 = f_4(\alpha, \beta, \gamma),$$

if for x, y, z, t of equations (1) we write y_1, y_2, y_3, y_4.

Conversely, a two-spread of order N in an $(N+1)$-flat is unicursal and every N-flat section of it is unicursal. For by successive projections from points on the spread it is projected into a two-spread of order $N-1$ in an N-flat, of order $N-2$ in an $(N-1)$-flat ...., of order 2 in a 3-flat and then into a plane. In like manner a curve of order N (in which the two-spread is cut by an N-flat R_N) in an N-flat may be projected into a line.*

Thus a two-spread of order N in an $(N+1)$-flat is itself unicursal and every flat-section of it is unicursal; and clearly, however projected, the spread retains these characteristics; in particular, it is projected from an $(N-3)$-flat not intersecting it upon a three-flat into a unicursal surface of order N every plane-section of which is unicursal.

The 1.1 *Correspondence between a Two-spread of Order N in an* $(N+1)$*-flat and a Plane.*

$y_1, y_2, \ldots. y_{N+2}$ are the homogeneous point-coordinates in the $(N+1)$-flat R_{N+1}. Choose $N-1$ points, $A_1, A_2, \ldots. A_{N-1}$, of the spread at random; that is, so that the $(N-2)$-flat passing through them meets the spread in no other point.

Let $\qquad L' \equiv \Sigma l'_\kappa y_\kappa = 0, \quad L'' \equiv \Sigma l''_\kappa y_\kappa = 0, \quad L''' \equiv \Sigma l'''_\kappa y_\kappa = 0$

be three asyzygetic N-flats through these points A. Then

(8) $\qquad \beta L' - \alpha L'' = 0, \qquad$ (9) $\quad \gamma L' - \alpha L''' = 0$

is an R_{N-1} meeting the spread in N points; that is, in the $N-1$ fixed points A and in one other point P depending on the ratios $\alpha:\beta:\gamma$. In fact, by a suitable choice of coordinate axes of α, β, γ in a fixed plane in the $(N+1)$-flat, the point of intersection P' of the R_{N-1} with this plane will have the homo-

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* Clifford: "On the Classification of Loci," Mathematical Papers, pp. 305-331.

geneous coordinates α, β, γ. This is the 1.1 correspondence by projection between the points of the two-spread and those of a plane.

Consider the system of equations formed by joining to (8) and (9) the equations of the two-spread (equivalent to $N-1$ independent equations) and the equation of a variable N-flat

$$u_y \equiv u_1 y_1 + u_2 y_2 + \ldots + u_{N+2} y_{N+2} = 0. \tag{10}$$

From these equations—one more than sufficient to determine $y_1, \ldots y_{N+2}$—we may eliminate $y_1, \ldots y_{N+2}$. The eliminant equated to zero is the condition that the N-flat $u_y = 0$ pass through some one of the N points common to (8), (9) and the spread, is in fact the tangential equation of these N points, and so is of degree N in the u. The eliminant is of degree N in α, β, γ also, because, considering $\alpha : \beta$ and the u as constant, there will be N values of $\alpha : \gamma$ which will make the equations consistent, or, what is the same thing, make the eliminant vanish, viz., the N values for every one of which the N-flat $\gamma L' - \alpha L''' = 0$ passes through one of the N points common to $\beta L' - \alpha L'' = 0$, the spread and the N-flat $u_y = 0$.

The eliminant is then a homogeneous function of the N^{th} degree of the u, and also of α, β, γ. Considered as a function of the u and equated to zero, it is the tangential equation of the N points common to (8), (9) and the spread; *i. e.*, of the $N-1$ fixed points A and of the one variable point P depending upon $\alpha : \beta : \gamma$. Hence the eliminant may be separated into the product of $N-1$ factors linear in the u and independent of α, β, γ (the tangential expressions for the $N-1$ fixed points A), and one factor linear in the u and of degree N in α, β, γ (the tangential expression of the point P). From this last factor we have the tangential equation of P,

$$u_1 F_1(\alpha, \beta, \gamma) + u_2 F_2(\alpha, \beta, \gamma) + \ldots + u_{N+2} F_{N+2}(\alpha, \beta, \gamma) = 0, \tag{11}$$

which must be identical with

$$u_1 y_1 + u_2 y_2 + \ldots + u_{N+2} y_{N+2} = 0, \tag{12}$$

where the y are the coordinates of point P.

Hence the coordinates of a point P of the spread are proportional to homogeneous functions of degree N of the coordinates of P' the projection of P on the (α, β, γ)-plane; that is,

$$y_\kappa = \rho F_\kappa(\alpha, \beta, \gamma), \qquad (\kappa = 1, 2, \ldots N+2), \tag{13}$$

where ρ is an arbitrary constant, the proportion-factor.

There is a 1.1 correspondence between the points on the *unicursal* curve of intersection of the spread with the N-flat R_N,

(14) $$a_1 y_1 + a_2 y_2 + \ldots + a_{N+2} y_{N+2} = 0,$$

and the points of the curve of order N in the correspondence-plane

(15) $$a_1 F_1(\alpha, \beta, \gamma) + a_2 F_2(\alpha, \beta, \gamma) + \ldots + a_{N+2} F_{N+2}(\alpha, \beta, \gamma) = 0.$$

Hence this curve, in fact every curve of the system determined by the $N+2$ fundamental curves $F_\kappa(\alpha, \beta, \gamma) = 0$, must be unicursal. Two curves of this system must meet in and only in N points (aside from the points common to all curves of the system), for these points are to correspond to the N points where the spread is intersected by the R_{N-1} of intersection of the two R_N corresponding to the two curves. In R_{N+1} the system of R_N is $(N+1)$-ply infinite, hence the corresponding system of curves must be $(N+1)$-ply infinite.

Suppose the system of curves of order N has α_r common r-ple points (for the moment assumed to be without further singularity). The intersection- and unicursality-equations may be written

(I) $$\Sigma r^2 \alpha_r = N^2 - N,$$

(II) $$\Sigma \frac{1}{2} r(r-1)\alpha_r = \frac{1}{2}(N-1)(N-2).$$

The base-point system cannot contain two multiple points the sum of whose orders is greater than N; otherwise *every* curve of the system would *break up* into the line joining the two multiple points and a supplementary curve, which is impossible. Hence, either all the α_k for $k > \frac{N}{2}$ equal 0, or if one equals unity all the remaining ones equal 0.

From (II) we have what is the same thing,

(II) $$\Sigma r(r-1)\alpha_r = N^2 - 3N + 2 = (N-1)(N-2),$$

which, subtracted from (I), gives

(III) $$\Sigma r \alpha_r = 2N - 2 = 2(N-1).$$

There is a general, say *the general*, solution,

$$\alpha_{N-1} = 1, \quad \alpha_1 = N-1, \quad \alpha_r = 0 \qquad (N-1 > r > 1).$$

There is no other solution with a multiple point of order $> \frac{N}{2}$, say of order $N-s$, where $1 < s < \frac{N}{2}$. For, suppose there were such a multiple point, the multiple point of next highest order might be of order s (so that sum of orders shall just equal but not exceed N).

In (II), (III) substitute for a_{N-s} 1, and for α_r $(s < r \mp N - s)$ 0, and expose outside the summation signs the leading terms, and we have

(IIa) $$s(s-1)\alpha_s + \Sigma t(t-1)\alpha_t = (s-1)(2N-s-2),$$
$$t < s,$$

(IIIa) $$s\alpha_s + \Sigma t\alpha_t = N + s - 2.$$

Multiplying (IIIa) by $s-1$ and subtracting (IIa), one has

(IVa) $$(s-1)\Sigma t\alpha_t - \Sigma t(t-1)\alpha_t = (s-1)(-N+2s).$$

Now since $s > 1$ and also $s > t$, the left side is positive or zero, while since $s < \frac{N}{2}$, the right side is negative. That is, the assumption that there is a solution—in addition to the general solution—with a multiple point of an order $> \frac{N}{2}$ leads to an absurd equation.

Any other solution must then have the order of the multiple point on highest order, say s, $\overline{\overline{<}} \frac{1}{2}N$. The equations become, where $t < s$,

(IIb) $$s(s-1)\alpha_s + \Sigma t(t-1)\alpha_t = (N-1)(N-2),$$

(IIIb) $$s\alpha_s + \Sigma t\alpha_t = 2(N-1).$$

Multiplying (IIIb) by $s-1$ and subtracting (IIb), one has

(IVb) $$(s-1)\Sigma t\alpha_t - \Sigma t(t-1)\alpha_t = (N-1)(2s-N).$$

Here the right side is negative when $s < \frac{1}{2}N$,

or zero when $$s = \frac{1}{2}N,$$

and not positive, since by hypothesis $s \overline{\overline{<}} \frac{1}{2}N$.

The left side $(t < s)$ is positive unless either $s = 1$, and $\therefore t = 0$, or $\alpha_t = 0$ for every $t < s$, when it is zero.

Thus there are only two possibilities: $s = \frac{1}{2}N$ and either $s = 1$ or ($s > 1$, but) $\alpha_t = 0$ for every $t < s$. In the first case,

$$s = 1 = \frac{1}{2}N, \therefore N = 2.$$

From (IIIb) $$s\alpha_s = \alpha_1 = 2(N-1) = 2.$$

This is, however, in fact, *the general* solution for $N = 2$,

$$\alpha_1 \textit{ quâ } \alpha_{N-1} = 1 \text{ and } \alpha_1 \textit{ quâ } \alpha_1 = N - 1 = 1; \text{ or } \alpha_1 = 2.$$

In the second case,

$$s = \frac{1}{2}N \text{ and } \alpha_t = 0 \text{ for every } t < s.$$

Equation (IIIb), with which (IIb) in view of (IVb) is identical, becomes

$$s\alpha_s = 2(N-1) = 4s - 2,$$
$$s(4-\alpha_s) = 2.$$

Since $s > 1$, $s = 2$, $\therefore 4 - \alpha_s = 4 - \alpha_2 = 1$, $\therefore \alpha_2 = 3$, and $N = 2s = 4$.
This exceptional solution, $N = 4$, $\alpha_2 = 3$, gives the 1.1 correspondence between an exceptional two-spread of order $N = 4$ in five-flat R_5 and the (α, β, γ)-plane,

(16) $$y_\kappa = \rho F_\kappa(\alpha, \beta, \gamma), \qquad (\kappa = 1, 2, \ldots . 6),$$

where the F are homogeneous functions of α, β, γ of fourth degree, and such that

(17) $$a_1F_1(\alpha, \beta, \gamma) + a_2F_2(\alpha, \beta, \gamma) + \ldots . + a_6F_6(\alpha, \beta, \gamma) = 0$$

is the equation of the 5-ply infinite system of unicursal quartic curves through three double points. By a quadratic transformation in the (α, β, γ)-plane of which the three double points are the fundamental points, this system of quartic curves transforms into the general (5-ply infinite) system of plane conics, say

(18) $$F_\kappa(\alpha, \beta, \gamma) = \rho' G_\kappa(\alpha', \beta', \gamma'), \qquad (\kappa = 1, 2, \ldots . 6),$$

where the G are homogeneous functions of α', β', γ' (the new coordinates) of the second degree, and ρ' is a proportion-factor. Thus the correspondence is given by

(19) $$y_\kappa = \rho'' G_\kappa(\alpha', \beta', \gamma'), \quad (\kappa = 1, 2, \ldots . 6), \ (\rho'' = \rho\rho').$$

I call this spread *Steiner's quartic two-spread in a space of five dimensions.* For it projects by the planes passing through a fixed line on a fixed three-flat into an ordinary Steiner's quartic in three dimensions. The coordinate-system $y_1, y_2, \ldots . y_6$ is as yet perfectly general. We may specialize by taking the coordinate-vertices 5, 6 on the line from which we project, and by taking for $y_5 = 0$ and $y_6 = 0$ two R_4 passing through the R_3 on which we project. Then the two-spread will be projected into a unicursal surface in the R_3 whose point for point correspondence with a plane is given, after suitable choice of planes of reference, by the equations

(20) $$y_\kappa = \rho'' G_\kappa(\alpha', \beta', \gamma'), \qquad (\kappa = 1, 2, 3, 4);$$

or, in fact, Steiner's quartic.

The general solution

$$\alpha_{N-1} = 1, \quad \alpha_1 = N - 1$$

gives the 1.1 correspondence between a two-spread of order N in an $(N+1)$-flat and the (α, β, γ)-plane,

(21) $$y_\kappa = \rho F_\kappa(\alpha, \beta, \gamma), \qquad (\kappa = 1, 2, \ldots . N+2),$$

where the F are homogeneous functions of α, β, γ of degree N, and such that

(22) $$a_1F_1(\alpha, \beta, \gamma) + a_2F_2(\alpha, \beta, \gamma) + \ldots + a_{N+2}F_{N+2}(\alpha, \beta, \gamma) = 0$$

is the equation of the $(N+1)$-ply infinite system of curves of order N through one $(N-1)$-ple point and $N-1$ simple points. A line through the $(N-1)$-ple point is met by a curve of the system in $N-(N-1)=1$ effective point. The curve on the two-spread corresponding to this line must then itself be a line, being met in only one point by the R_N meeting the two-spread in the curve corresponding to the curve of the plane-system. That is, all two-spreads of order N in an $(N+1)$-flat are *ruled*.

It is now necessary to remove the restriction in singularity made with reference to the common multiple points of the system of curves of order N in the (α, β, γ)-plane. Clebsch (Vorlesungen über Geometrie, pp. 491–496) has shown how an i-ple point, as complicated as may be, may be considered as an i-ple point which has absorbed a certain number of other multiple points of definite order; of these, every k-ple point is equivalent (in questions relating to the class or the deficiency of the curve) to $\frac{1}{2}k(k-1)$ double-points of which a certain number are cusps. Now, from this standpoint, let γ_r denote the number of r-ple points common to all curves of the system (that is, $\gamma_r = \alpha_r + \beta_r$, where α_r is the number of explicit r-ple points common and β_r is the number of r-ple points absorbed by the various explicit multiple points common to all curves of the system); and let c denote the number of cusps. In the unicursality-equation a cusp plays the role of an ordinary double-point. In the intersection-equation a k-ple point (*whether explicit or absorbed*), common to the two curves, counts for a number of intersections equal to $k^2 +$ the number of cusps it contains + the number of intersections in multiple points absorbed by it. (Clebsch, to be sure, discusses only one curve, and not at all the intersection of two curves; but the truth of the statement made will appear from the discussion of Clebsch, when one bears in mind that in an ordinary quadratic transformation with a fundamental point at the k-ple point common to the two curves, k^2 of the intersections at the k-ple point disappear, but all others remain as intersections of the transformed curves.) Thus the base-point system is equivalent to $\Sigma r^2\gamma_r + c$ intersections of two arbitrary curves of the system.

The general intersection- and unicursality-equations are then

(I) $$\Sigma r^2\gamma_r + c = N^2 - N,$$

(II) $$\Sigma \frac{1}{2} r(r-1)\gamma_r = \frac{1}{2}(N-1)(N-2),$$

whence

(III) $$\Sigma r\gamma_r + c = 2N - 2.$$

The general solution is

$$\gamma_{N-1} = 1, \quad \gamma_1 + c = N - 1.$$

The base-point system cannot contain two multiple points (explicit or absorbed) the sum of whose orders exceeds N; for if so, every curve of the system would degenerate, which is not admissible. This will be proved presently; but with its aid we can show, as before, that aside from the general solution just given, there is the single exceptional solution corresponding to Steiner's quartic two-spread,

$$c = 0, \quad N = 4, \quad \gamma_2 = 3.$$

The equations (IVa), (IVb) are in this case, where the s and t have the same meanings as before,

(IVa) $$c(s-1) + (s-1)\Sigma t\gamma_t - \Sigma t(t-1)\gamma_t = (s-1)(-N+2s),$$

which is, as before, an absurdity;

(IVb) $$c(s-1) + (s-1)\Sigma t\gamma_t - \Sigma t(t-1)\gamma_t = (N-1)(2s-N),$$

which furnishes two possibilities: first,

$$s = 1, \quad N = 2, \quad c = 0,$$

since only multiple points contain cusps, $\gamma_1 = 2$, which is a particular case of the general solution; second,

$$s > 1, \quad s = \frac{1}{2}N, \quad c = 0, \quad s = 2, \quad N = 4, \quad \gamma_2 = 3,$$

which corresponds to a Steiner's quartic two-spread as previously defined, as will be shown later.

We return to the proof that if the sum of the orders of two multiple points (explicit or absorbed) on a curve of order N exceeds N, the curve degenerates. The order of an absorbed multiple point is not greater than that of the absorbing multiple point. The theorem is evident, then, except in the case where an explicit multiple point, say of order i, absorbs an l_1-ple point where $i + l_1 > N$. By reference to Clebsch (Vorlesungen ü. Geom., p. 493), whose notation we use, such an absorption occurs as follows:

Let the i-ple point be at $x_1 = x_2 = 0$, where l tangents ($l \leqq i$) fall together in the line $k_1 x_1 + k_2 x_2 = 0$, say $\alpha = 0$. The equation of the curve may be written

$$C \equiv g_{i-l}(x_1, x_2)\, \alpha^l x_3^{N-i} + f_{i+1}(x_1, x_2)\, x_3^{N-i-1} + \ldots + f_{i+t}(x_1, x_2)\, x_3^{N-i-t} + \ldots + f_N(x_1, x_2) = 0,$$

where the f, g denote homogeneous functions of the arguments of degree equal to the subscript-numeral.

Now let $f_{i+r}(x_1, x_2)$ for $r = 1 \ldots l_1$ (where $l_1 \leqq l \leqq i$) contain α as a factor at least $l_1 - r$ times. Then the equation may be written

$$C \equiv g_{i-l}(x_1, x_2)\, \alpha^l x_3^{N-i} + \alpha^{l_1 - 1} g_{i-l_1+2}(x_1, x_2)\, x_3^{N-i-1} + \ldots + \alpha^{l_1 - r} g_{i-l_1+2r}(x_1, x_2) x_3^{N-i-r} + \text{etc.} = 0.$$

This i-ple point (cf. Clebsch, l. c.) has absorbed an l_1-ple point (and also, with which we are not at present concerned, $l - l_1$ cusps). Now, the hypothesis of our theorem is that

$$i + l_1 > N, \quad l_1 > N - i,$$

and the last term on the left in the equation of the curve is, for $r = N - i$ (in fact $< l_1$),

$$\alpha^{l_1 - (N-i)} g_{i-l_1+2(N-i)}(x_1, x_2) = \alpha^{l_1 + i - N} g_{2N-l_1-i}(x_1, x_2),$$

and in fact the curve breaks up, consisting of the line α taken $l_1 + i - N$ times and a supplementary curve.

Corresponding to the general solution

$$\gamma_{N-1} = 1, \quad \gamma_1 + c = N - 1,$$

there is, as before, a unicursal ruled two-spread of order N in $(N+1)$-flat.

The exceptional solution

$$N = 4, \quad \gamma_2 = \alpha_2 + \beta_2 = 3$$

has three cases.

(A). $\alpha_2 = 3$, $\beta_2 = 0$. Three explicit double points. This is the case previously discussed which led to the definition of Steiner's quartic two-spread in a five-flat.

(B). $\alpha_2 = 2$, $\beta_2 = 1$. Two explicit double points, say (x_1, x_2), (x_2, x_3), of which one, say (x_1, x_2), has absorbed a third double point along the line x_1. The equation of the system of quartic curves is

$$C \equiv a x_1^2 x_3^2 + x_1 x_2 x_3 (b_1 x_1 + b_2 x_2) + x_2^2 (c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2) = 0.$$

By the quadratic transformation,

$$\left.\begin{array}{l} x_1 : x_2 : x_3 = z_2^2 : z_1 z_2 : z_1 z_3 \\ z_1 : z_2 : z_3 = x_2^2 : x_1 x_2 : x_1 x_3 \end{array}\right\}$$

which has a fundamental point at (x_2, x_3) and two coincident ones at (x_1, x_2) along the line x_1, this system of quartic curves transforms into

$$C' \equiv a z_3^2 + b_1 z_2 z_3 + b_2 z_1 z_3 + c_1 z_2^2 + c_2 z_1 z_2 + c_3 z_1^2;$$

that is, into the 5-ply infinite system of plane conics. Hence the two-spread given by case (B) is a Steiner's quartic two-spread as defined under case (A).

(C). $\alpha_2 = 1$, $\beta_2 = 2$. One explicit double point (x_1, x_2) which absorbs a second along the line x_2, which has in turn absorbed a third (along the conic $x_2 x_3 - x_1^2$). Endeavoring to construct the system of quartic curves having in common a singular point of this nature—after the discussion of Clebsch, a clear though rather long problem—we find that, by suitable choice of reference lines and constants, the equation may be written

$$C \equiv a\,(x_2 x_3 - x_1^2)^2 - b x_1 x_2 (x_2 x_3 - x_1)^2 + x_2^2 (c x_1^2 + d x_1 x_2 + e x_2^2 + f x_2 x_3) = 0.$$

By the quadratic transformation,

$$\left.\begin{array}{l} x_1 : x_2 : x_3 = z_1 z_2 : z_2^2 : z_2 z_3 + z_1^2 \\ z_1 : z_2 : z_3 = x_1 x_2 : x_2^2 : x_2 x_3 - x_1^2 \end{array}\right\}$$

which has three consecutive fundamental points at (x_1, x_2) along the conic $x_2 x_3 - x_1^2$, this system of quartic curves transforms into

$$C' \equiv a z_3^2 - b z_1 z_3 + c z_1^2 + d z_1 z_2 + e z_2^2 + f\,(z_2 z_3 + z_1^2) = 0;$$

that is, since the six quadratic expressions are asyzygetic, into the 5-ply infinite system of plane conics. Hence the two-spread given by case (C) is a Steiner's quartic two-spread as originally defined.

The principal results of the article may be collected in the following theorems:

An algebraic two-spread [two-dimensional surface] *of order N in a flat space of any number of dimensions of which every flat section is unicursal, is the projection of a two-spread of order N in an $(N+1)$-flat* [flat space of $N+1$ dimensions].

A two-spread of order N in an $(N+1)$-flat is unicursal and every N-flat section of it is unicursal.

All such two-spreads, with the exception of Steiner's quartic two-spread in a five-flat, are ruled.

Projections of these spreads have corresponding properties.

In a memoir by Veronese, "Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens" (Math. Annalen, 1882, XIX, pp. 161–234), in V, § 4, p. 224, *unicursal ruled* two-spreads of order N in an $(N+1)$-flat are considered, and further by the writer, "Extensions of certain Theorems of Clifford and of Cayley in the Geometry of n-Dimensions"* (Trans. Conn. Acad., VII, 1885), who proved there that *all* two-spreads of order N in an $(N+1)$-flat are *unicursal*, and in this article that all, with the exception of Steiner's quartic, are *ruled.*

EVANSTON, ILL., July 2, 1887.

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* I take this opportunity of saying that when the article "Extensions," etc., was written I had not seen the article of Veronese, and that my theorems A of I, p. 10, and 1 of IV, p. 24, are given by Veronese on p. 167 and p. 192 respectively, and further in a note, p. 228, he gives my first abbildung-system of p. 12.